VARIATIONAL STUDY OF NONLINEAR SPLINE CURVES

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Abstract

This is an exposition of the variational and differential properties of nonlinear spline curves, based on the Euler-Bernoulli theory for the bending of thin beams or elastica. For both open and closed splines through prescribed nodal points in the euclidean plane, various types of nodal constraints are considered, and the corresponding algebraic and differential equations relating curvature, angle, arc length, and tangential force are derived in a simple manner. The results for closed splines are apparently new, and they cannot be derived by the consideration of a constrained conservative system. There is a survey of the scanty recent literature.

Keywords: spline, elastica, curve-fitting, approximation, nonlinear, natural boundary conditions, variational, closed curve

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1. Linear and nonlinear splines. Let A be a finite ordered set of points in the euclidean plane, with Cartesian coordinates (x_r, y_r), $r = 1,\ldots,n-1$, through which it is desired to pass a smooth curve. An old technique in drafting is to use a mechanical spline to form a smooth curve C that contains A. In the present day of automatic plotters, numerically controlled milling machines, and so on, it is more important to find a mathematical or computational representation of a suitable C than to draw it. Thus one uses some mathematical model of the mechanical spline.

By far the most widely used model is a linear (cubic) spline, suitable when the curve C in some x-y coordinate system is the graph of a function f, so that $y = f(x)$, $x_0 \le x \le x_n$. Assume that $x_0 < x_1 < \ldots < x_n$. The linear spline can be defined as the unique function f for which

$$\int_{x_0}^{x_n} f''(x)^2 dx \qquad (1)$$

is minimized among all twice continuously differentiable functions assuming the value y_r at x_r $(r = 1,2,\ldots,n-1)$. (According to this definition, f will satisfy the natural end conditions $f''(x_0) = f''(x_n) = 0$. There are alternative treatments of the end conditions.)

The linear spline so defined turns out to be a (usually) different cubic polynomial in each interval (x_{r-1}, x_r) , with matching values, derivatives, and second derivatives (and hence curvatures) at each interior node x_r $(r = 1,2,\ldots,n-1)$. The spline will actually be a straight line segment for $x_0 \le x \le x_1$ and $x_{n-1} \le x \le x_n$.

The theory of linear splines has grown enormously in the last decade, and these curves and various linear generalizations have both practical and theoretical importance in the approximation of known functions, solutions of differential equations, and so on. The reader can find an exposition, with generalizations and applications, in Ahlberg, Nilson, and Walsh [1].

Linear splines are invariant under linear changes in the y-coordinate alone, as Podolsky and Denman [9] point out. Thus they are suited to such problems as the interpolation of data, where x and y have different meanings. On the other hand, linear splines are not invariant under rotations of the x-y coordinate system, and hence are not well. suited to the interpolation of geometrical points in the euclidean plane. Moreover, linear splines cannot be used directly to define a closed curve C in the X-Y plane.

For the purposes of interpolating points in the euclidean plane it is appropriate to find a mathematical model which is invariant under all similarity transformations. The model we treat is sometimes called the <u>elastica</u>, but we shall refer to it as a <u>nonlinear spline</u>. As a preliminary to work on actually computing nonlinear splines, we have investigated their precise definition, including variational properties, defining equations, continuity conditions, and end conditions, both for open and closed curves.

The term nonlinear spline is used variously in the literature: (a) If the integrand $f''(x)^2$ of (1) is multiplied by a nonconstant weight factor, sometimes the function that minimizes the altered problem is called a nonlinear spline. (b) Suppose one is given a function $\varphi(x)$ to be approximated by a linear cubic spline passing through n-1 points $(x_r, \varphi(x_r))$, and that the n-1 abscissas x_r are varied until the spline best approximates φ in some given norm. Sometimes the result is called a nonlinear spline. However, in both (a) and (b) above the splines satisfy a linear differential equation in each interval, whereas our nonlinear splines satisfy a nonlinear differential equation.

We do not claim that computing nonlinear splines will necessarily be an economical way to interpolate points in the x-y plane. Moreover, nonlinear splines are not invariant under linear changes in the y-coordinate alone, so that they seem ill-suited to the interpolation of data where x and y are unrelated.

We have been interested only in studying as carefully as we could the mathematical nature of these nonlinear splines. In this paper we present a variational treatment of nonlinear splines, emphasizing the natural boundary conditions of the problem. We believe that our treatment of the closed nonlinear spline may be new.

2. Previous work. In the theory of elasticity, our mathematical model of the mechanical spline is called a thin beam or elastica, and its treatment dates back to James and Daniel Bernoulli, Euler, Kirchhoff, and others. The history and theory are summarized by Love [7]. None of those treatments dealt directly with the use of the nonlinear spline to interpolate points, and there was little discussion of closed splines.

The earliest discussion that we have seen in print of the use of nonlinear splines for interpolation is that of Birkhoff and de Boor [2]. That paper refers to excellent laboratory reports by Fowler and Wilson [4] and by Birkhoff, Burchard, and Thomas [3]. Glass [5] briefly describes computations of open nonlinear splines in Cartesian coordinates. Hosaka [6] describes the generation of nonlinear splines on a digital differential analyzer. Woodford [12] describes an iterative procedure for interpolation with open nonlinear splines that is much faster than Glass's algorithm; he also works with Cartesian coordinates.

In his Ph.D. dissertation, Mehlum [8] discusses the nature of nonlinear open splines, again using a Cartesian coordinate system rotated to a convenient local orientation. He also gives an algorithm for computing an approximation to the nonlinear spline by a succession of circular arcs meeting with a continuous tangent but discontinuous curvature.

3. Basic concepts of bending theory of thin beams. Of all the curves that pass in turn through the ordered set A of points (x_r, y_r) mentioned in the introduction, we shall consider as admissible only those whose tangent direction is continuous everywhere, and whose curvature is piecewise continuous, with discontinuities in curvature permitted on any finite set of points. A plausible suggestion for the smoothest of these admissible curves is that the integral of the square of the curvature with respect to arc length should attain a minimum. This comprises a simple representation of the concept of a curve passing through the points with minimum total bend amplitude, and will be utilized in the form of the necessary condition .

$$\delta \int_{\ell_0}^{\ell_n} \kappa^2 ds = 0 \tag{2}$$

where κ is the curvature, s is arc length, $\ell_n - \ell_0$ is the total length of the curve, and δ is the symbol for variation. The integral in (2) is proportionalto the strain energy in a bent spline according to Euler-Bernoulli beam theory, and we show in Section 4 that (2) is the variational form of the conditions of equilibrium for the spline with forces applied only at the support points. It seems, therefore, appropriate to investigate spline interpolation in terms of mechanical bending theory, and it will be shown in the present paper that this approach does lead to the introduction of variable&which are particularly convenient for interpreting spline interpolation, and perhaps also for computing splines.

Bernoulli-Euler theory, as described in detail by Love [7], is the simplest form of beam theory, and considers only bending deformations, neglecting shear deformations and stretching of the center line of the beam. Such an approximation is satisfactory for beams with cross-section dimensions small compared to the span between supports, as clearly applies for splines. Such restricted deformations are introduced by requiring that plane sections normal to the center line in the undeformed state remain plane and normal to the deformed center line, and that the center line of the beam does not stretch.

The forces and moments on a beam element are shown in Figure 1, where M is the bending moment, S is the shearing force, and P is the longitudinal force. (The convention is $P > 0$ for tension, and $P < 0$ for compression.) The assumptions about deformation mentioned above, combined with Hooke's law relating stress and strain, yield

$$M = EI\kappa \, , \tag{3}$$

where E is Young's modulus of elasticity, I is the second moment of the section about the axis of bending,

$$\kappa = \frac{d\theta}{ds} \, , \tag{4}$$

and θ is the angle between the beam and the x-axis. The equations of equilibrium for each unloaded span between supports, which are deduced in Section 4, are as follows:

for moments:

$$\frac{dM}{ds} + S = 0 \, ; \tag{5}$$

for normal forces:

$$\frac{dS}{ds} + P\kappa = 0 \, ; \tag{6}$$

for longitudinal forces:

$$\frac{dP}{ds} - S\kappa = 0 \, . \tag{7}$$

It is convenient to work in terms of reduced force variables:

$$\bar{M} = \frac{M}{EI} = \kappa \, , \quad \bar{S} = \frac{S}{EI} \, , \quad \bar{P} = \frac{P}{EI} \, . \tag{8}$$

Then (5) and (6) give:

$$\bar{S} = -\frac{d\kappa}{ds} \, , \tag{9}$$

$$\bar{P} = \frac{1}{\kappa}\frac{d^2\kappa}{ds^2} \, , \tag{10}$$

and (7) becomes

$$\frac{d}{ds}\left[\frac{1}{\kappa}\frac{d^2\kappa}{ds^2}+\frac{\kappa^2}{2}\right] = 0 \quad , \tag{11}$$

or

$$\frac{1}{\kappa}\frac{d^2\kappa}{ds^2}+\frac{\kappa^2}{2} = \bar{P} + \frac{\kappa^2}{2} = c_{r-1} \quad , \tag{12}$$

where c_{r-1} is a constant of integration for the r-th span.

We must now consider the boundary conditions at the supports which constrain the spline to pass through the required points Q_r of A . The least constraining such support is a freely rotating sleeve attached to the point Q_r that permits free rotation of the spline and free sliding through the sleeve. The only support force is therefore normal to the sleeve, and this does no work on a possible motion of the spline through the sleeve. A more constraining support would be a pin through the spline which permits free rotation but no sliding, or a pin with rotation prevented. In none of these cases is work done by support forces, since either a force (or moment) component is zero, or the associated motion is zero, and such supports are termed **workless** constraints.

Figure 2 shows a spline passing through freely rotating, sliding sleeves at $Q_1,\ldots,Q_{n-1}$, where Q_0 and Q_n are the free ends of the spline. The configuration of the spline could be analyzed using the equations given above, but a simpler and more revealing approach for our purposes is to observe that this spline forms a conservative mechanical system with potential energy given by the strain energy of the spline,

$$U = \int_{\ell_0}^{\ell_n} (EI\kappa^2/2)\,ds \quad ; \tag{13}$$

there are no other contributions to U , since the external forces are all workless. The theory of conservative systems [11] tells us that at a stable equilibrium configuration of the spline, the energy (13) is a local minimum, which implies (2) for a uniform spline with EI constant. Moreover, any constraint added to the system, such as changing a freely sliding sleeve to a pin support that prevents sliding, will either increase the potential energy in the corresponding equilibrium configuration, or leave it unchanged if the added constraint happens to be compatible with the configuration. Thus

$$\int_{a}^{\ell_n} \kappa^2 ds \tag{14}$$

will also exhibit a local minimum in the configuration shown in Figure 2, relative to variations of the constraints. Note that the free ends Q_0 and Q_n , with no forces or moments applied, also provide workless boundary conditions, and any constraint on their freedom of motion will increase the energy expression (14). Thus a local minimum of the integral (14) corresponds to free ends and freely rotating sliding constraints at $Q_1, \ldots, Q_{n-1}$. This cannot be a global minimum in the space of all configurations, since a lower value of the integral in (14) can be achieved, as pointed out in [2], by introducing large loops between supports, which, of course, modify the topology.

In the next two sections we deduce the least constraining support conditions for the spline passing through the points $Q_1, \ldots, Q_{n-1}$ by seeking the minimum of the integral (14) directly through analysis of the variational problem (2), and deduce the natural boundary conditions that yield this minimum energy configuration. Although this approach simply reproduces the minimum constraint conditions shown in Figure 2, and anticipated above on the basis of conservative system theory, it is independently useful, since it permits investigation of the closed spline problem in Section 6. The latter problem cannot be treated directly by the theory of a constrained conservative system, because we must consider the effect of variable arc length for the closed curve, and this changes the system more than simply by imposing a constraint.

4. <u>Deductions from the variational statement</u>. We consider the variational statement (2) with integration limits ℓ_0 and ℓ_n for a curve constrained to pass through the points $Q_1,\ldots,Q_{n-1}$ with end points Q_0 and Q_n. The points $Q_0,\ldots,Q_n$ correspond to values $\ell_0,\ldots,\ell_n$ of the arc length s. In the present section we do not consider end conditions. Thus we do not care whether the curve is open (as in Figure 2) or closed (Q_0 and Q_n coincide). We shall prove in this section that if such a curve satisfies (2) -- and is hence a spline in our sense -- then the spline is the position of a thin beam satisfying equations (3) -(7) of Section 3.

Because of the constraints, (2) takes the form

$$\delta \sum_{r=1}^{n} \int_{\ell_{r-1}}^{\ell_r} \kappa^2 ds = 0 . \tag{15}$$

The fact that the spline passes through the points $Q_0,\ldots,Q_n$ prescribes the following constraint conditions for $r = 1,\ldots,n$:

$$\int_{a_{r-1}}^{\ell_r} \cos\theta \, ds - x_r + x_{r-1} = 0 \, , \tag{16a}$$

$$\int_{a_{r-1}}^{\ell_r} \sin\theta \, ds - y_r + y_{r-1} = 0 \, , \tag{16b}$$

where Q_r has the coordinates (x_r, y_r). Note in (16a), (16b) that x_r and y_r are prescribed numbers for $r = 1,\ldots,n-1$, whereas x_0, y_0, x_n and y_n are free to vary.

We follow the standard techniques of the calculus of variations and introduce Lagrange multipliers λ_{r-1} and μ_{r-1} for (16a) and (16b), respectively $(r = 1,\ldots,n)$. We take care of the constraints (16a), (16b) by seeking a stationary value of the functional

$$\sum_{r=1}^{n} \int_{\ell_{r-1}}^{\ell_r} \kappa^2 \, ds + \sum_{r=1}^{n} \lambda_{r-1} \left[\int_{\ell_{r-1}}^{\ell_r} \cos\theta ds - x_r + x_{r-1} \right]$$

$$+ \sum_{r=1}^{n} \mu_{r-1} \left[\int_{a_{r-1}}^{\ell_r} \sin\theta ds - y_r + y_{r-1} \right]$$

$$= \sum_{r=1}^{n} \left[\int_{a_{r-1}}^{\ell_r} (\kappa^2 + \lambda_{r-1} \cos\theta + \mu_{r-1} \sin\theta) ds \right.$$

$$\left. + \lambda_{r-1}(x_{r-1} - x_r) + \mu_{r-1}(y_{r-1} - y_r) \right] \tag{17}$$

with respect to a general smooth variation $\delta\theta(s)$, and variations δx_0 , δy_0 , δx_n , δy_n , combined with sliding through the pivots

$$\delta s(Q_r) = \delta \ell_r \qquad (r = 1,\ldots,n-1) . \qquad (18)$$

Setting the variation of (17) to zero and integrating by pa&s,*/ we get the form

$$\sum_{r=1}^{n} \Bigg\{ 2\kappa\, \delta\theta \Bigg|_{\ell_{r-1}^{+}}^{\ell_r^{-}} + \int_{\ell_{r-1}}^{\ell_r} [-2\frac{d\kappa}{ds} - \lambda_{r-1}\sin\theta + \mu_{r-1}\cos\theta]\delta\theta\, ds$$

$$+ [\kappa^2 + \lambda_{r-1}\cos\theta + \mu_{r-1}\sin\theta]_{\ell_r^{-}}\ \delta\ell_r$$

$$- [\kappa^2 + \lambda_{r-1}\cos\theta + \mu_{r-1}\sin\theta]_{\ell_{r-1}^{+}}\ \delta\ell_{r-1} \Bigg\}$$

$$+ \lambda_0\, \delta x_0 + \mu_0 \delta y_0 - \lambda_{n-1}\, \delta x_n - \mu_{n-1}\, \delta y_n = 0 . \qquad (19)$$

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*/ In integrating by parts, we assume that the curvature $\kappa(s)$ of the minimizing curve is continuously differentiable in each interval $\ell_{r-1} < s < \ell_r$. If the curvature $\kappa(s)$ of the minimizing function is assumed only to be piecewise continuous, but e(s) is continuous, then it can be proved by a different argument based on a lemma of du Bois-Reymond that $\kappa(s)$ is in fact continuously differentiable in each interval. This justifies our introduction of the broad class of admissible curves at the start of Section 3.

By ℓ_r^{+}, ℓ_r^{-} in the following we mean the limiting values ℓ_r+0 and ℓ_r-0 .

The integral term of (19) yields for $r = 1,\ldots,n$:

$$-2\frac{d\kappa}{ds} - \lambda_{r-1}\sin\theta + \mu_{r-1}\cos\theta = 0\ ,\quad \ell_{r-1} < s < \ell_r\ , \tag{20}$$

which can be integrated, using equations analogous to (16a) and (16b) for an open interval, giving

$$\kappa(s) = \kappa(\ell^{+}_{r-1}) - \frac{\lambda_{r-1}}{2}(y - y_{r-1}) + \frac{\mu_{r-1}}{2}(x - x_{r-1})\ ,\ (r = 1,\ldots,n)\ . \tag{21}$$

Identifying κ with $\bar{M}$, as in (8), we see that (21) comprises a moment relation for the part of the spline between the arc lengths ℓ_{r-1} and s , as illustrated in Figure 3. Thus the Lagrange multiplier faators $\lambda_{r-1}/2$ and $\mu_{r-1}/2$ are simply the force components acting on the spline at ℓ^{+}_{r-1} $(r = 1,\ldots,n)$. By equilibrium considerations, these same force components can be considered to act on any section of the spline with $\ell_{r-1} < s < \ell_r$, so that, taking components along and normal to the spline, the tensile force $\bar{P}$ and shear force $\bar{S}$ are given for $\ell_{r-1} < s < \ell_r$ by

$$\bar{P} = -\frac{\lambda_{r-1}}{2}\cos\theta - \frac{\mu_{r-1}}{2}\sin\theta\ , \tag{22a}$$

$$\bar{S} = \frac{\lambda_{r-1}}{2}\sin\theta - \frac{\mu_{r-1}}{2}\cos\theta\ . \tag{22b}$$

Differential equation (20) can be alternatively integrated by writing

$$\frac{d\kappa}{ds} = \frac{d\kappa}{d\theta}\cdot\frac{d\theta}{ds} = \kappa\frac{d\kappa}{d\theta}\ ,$$

whence, in view of (22a),

$$\frac{\kappa^2}{2} + \bar{P} = c_{r-1} \qquad (\ell_{r-1} < s < \ell_r)\ , \tag{23}$$

where c_{r-1} is an integration constant.

Note that (20) and (22b) yield (9). Differentiating (20) with respect to s and using (22a) give (10). Finally, (12) and (11) follow from (23). The basic equations (5), (6), and (7) simply express (9) - (12) in different variables,.. and hence the equilibrium equations (5) - (7) are consequences of the variational statement (2).

It could conversely be proved that the satisfaction of equations (5), (6), and (7) implies that the variational condition (2) holds. Thus the variational condition (2) and the thin beam equations (5) - (7) provide equivalent foundations for the theory of nonlinear splines.

Since $\delta\theta$ is a continuous variation,

$$\delta\theta(\ell_r^-) = \delta\theta(\ell_r^+) \qquad (r = 1,\ldots,n-1) \ , \tag{24}$$

and the first term of (19) then demands that

$$\kappa(\ell_r^-) = \kappa(\ell_r^+) \qquad (r = 1,\ldots,n-1) \ . \tag{25}$$

In view of (18), (22a), and the terms in (19) containing $\delta\ell_r$, we then find that

$$\bar{P}(\ell_r^-) = \bar{P}(\ell_r^+) \qquad (r = 1,\ldots,n-1) \ . \tag{26}$$

5. <u>The open spline</u>. For the configuration shown in Figure 2 with free ends, δx_0 , δy_0 , $\delta\theta(\ell_0)$, δx_n , δy_n , $\delta\theta(\ell_n)$ are arbitrary variations. Hence the first and last terms in (19) demand that

$$\kappa(\ell_0) = \kappa(\ell_n) = \lambda_0 = \mu_0 = \lambda_{n-1} = \mu_{n-1} = 0 \ . \tag{27}$$

Thus from (22a, 22b) the end conditions became

$$\kappa(\ell_0) = \bar{P}(\ell_0) = \bar{S}(\ell_0) = \kappa(\ell_n) = \bar{P}(\ell_n) = \bar{S}(\ell_n) = 0 \ . \tag{28}$$

Thus the variational condition (15) implies that the open spline satisfies the natural boundary conditions (25), (26), (28), which are precisely the conditions associated with the least constraining supports depicted in Figure 2 and discussed in Section 3. In view of these relations, (23) holds for the entire spline $\ell_0 \le s \le \ell_n$ with $c_{r-1} = 0$ for all r :

$$\frac{\kappa^2}{2} + \bar{P} = 0 \quad , \tag{29}$$

and hence the differential equation

$$\frac{d^2\kappa}{ds^2} + \frac{\kappa^3}{2} = 0 \tag{30}$$

is valid as a special case of (12) for the threaded spline with free ends. This equation has been given by Birkhoff et. al. in [3]. Note that, in view of (25), (30) requires $d^2\kappa/ds^2$ to be continuous across supports, although in general $d\kappa/ds$ is discontinuous, because the lateral support force changes the shear force $\bar{S}$, which satisfies (9).

We wish to emphasize that our equations apply to any spline curve that satisfies the constraints of the problem, no matter what its topology. As is pointed out in [2], there may be sets of nodes A for which no spline exists and, if any spline exists for A , there may exist others satisfying the same constraints, with different numbers of loops between some adjacent pair of nodes. We know of no theorems about the existence or uniqueness of solutions to these problems.

6. <u>Closed nonlinear splines</u>. Now consider fitting a smooth <u>closed</u> curve through a set of prescribed points. We will express this situation by utilizing the previous development, but requiring that the points Q_0 and Q_n be coincident at an n-th prescribed point, and that the tangent to the curve be continuously turning also through that point. Thus, for some integer m related to the number of loops in the curve,

$$x_0 = x_n , \quad y_0 = y_n , \quad \theta_n = \theta_0 + 2m\pi ; \tag{31}$$

$$\left.\begin{aligned} &\sum_{r=1}^{n} \int_{\ell_{r-1}}^{\ell_r} \cos\theta \, ds = \sum_{r=1}^{n} \int_{\ell_{r-1}}^{\ell_r} \sin\theta \, ds = 0 ; \\ &\sum_{r=1}^{n} \int_{\ell_{r-1}}^{\ell_r} \kappa \, ds = 2m\pi . \end{aligned}\right\} \tag{32}$$

The deductions from (19) are unchanged from those described heretofore, apart from the contributions at $s = \ell_0$ and $s = \ell_n$. To obtain a local minimum of the integral (14), in order to find a "smoothest" closed curve through the n prescribed points, we must compare curves of slightly different total arc length, and this can be achieved by selecting the variations $\delta\ell_0$ and $\delta\ell_n$ to be unequal. Since the tangent to the curve prior to the variation is continuously turning, and that after the variation must also be, the variation8 at ℓ_0 and ℓ_n must satisfy

$$\delta\theta(\ell_0) + \kappa(\ell_0)\delta\ell_0 = \delta\theta(\ell_n) + \kappa(\ell_n)\delta\ell_n \quad . \tag{33}$$

It is not correct to demand that $\delta\theta(\ell_0) = \delta\theta(\ell_n)$, since elements of curved arc have been inserted into the loop in superposing the variation. Since $\delta\ell_0$, $\delta\ell_n$, $\delta\theta(\ell_0)$ and $\delta\theta(\ell_n)$ are no longer independent, , the terms arising from these variations in (19) must be combined with (33) to deduce the natural boundary conditions at the support $Q_0 = Q_n$. At the boundaries ℓ_0 and ℓ_n , (19) and (22a) give:

$$2[\kappa(\ell_n)\delta\theta(\ell_n) - \kappa(\ell_0)\delta\theta(\ell_0)] + [\kappa^2(\ell_n) - 2\bar{P}(\ell_n)]\delta\ell_n$$
$$-[\kappa^2(\ell_0) - 2\bar{P}(\ell_0)]\delta\ell_0 = 0 \quad . \tag{34}$$

Eliminating $\delta\theta(\ell_0)$ from (33) and (34) gives:

$$[\kappa^2(\ell_0) + 2\bar{P}(\ell_0)]\delta\ell_0 + 2[\kappa(\ell_n) - \kappa(\ell_0)]\delta\theta(\ell_n)$$
$$+ [\kappa^2(\ell_n) - 2\kappa(\ell_0)\kappa(\ell_n) - 2\bar{P}(\ell_n)]\delta\ell_n = 0 \,, \tag{35}$$

where the variations $\delta\ell_0$, $\delta\theta(\ell_n)$ and $\delta\ell_n$ can now be considered arbitrary and independent. Thus

$$\kappa(\ell_0) = \kappa(\ell_n) \tag{36}$$

and

$$(\kappa^2 + 2\bar{P})_{\ell_0} = (\kappa^2 + 2\bar{P})_{\ell_n} = 0 \quad . \tag{37}$$

Thus (29) and (30) again apply throughout the spline. Hence the natural boundary conditions for (15) yield the same integration constant $c_{r-1} = 0$

in (12) for the closed spline as for the free-ended open one. However, for the closed spline, this result does not follow from the least-constraint discussion of conservative systems. In fact, either adding or removing an element of arc from the optimum configuration increases the strain energy at equilibrium and hence exhibits this property associated with imposing additional constraint.

7. Comments and examples. When curve fitting with smooth curves is investigated, the variational principle (15), utilizing natural boundary conditions, calls for continuity of κ and $\bar{P}$ across supports, as well as the continuity of θ prescribed in the formulation of the problem. Geometrical discussions of the problem commonly take into consideration only continuity of θ and κ, but this seeming omission of $\bar{P}$ is in fact automatically taken care of by the differential equation (30), satisfied by the spline in each span between supports, since (30) and (29) are synonymous.

The variational principle (2) will yield (23) and (10), and hence the differential equations (12) or (11), for types of support other than the least constraining one treated in Section 4 above. These include, for example, pin supports which prevent sliding, built-in supports which prevent both displacement and rotation, and a fixed-angle freely displacing constraint. In general, with such supports, the constants c_r in (23) will not be zero, and will change from span to span along the spline, so that the differential equations (12) or (11) govern the deflection of the spline spans, and not the special case (30). These comprise the more general elastica curves discussed in [7], for which applied forces are not all acting in the direction of the normal to the spline at the point of application, or for which, in the closed spline case, the spline does not have the optimum length corresponding to (37). Note that in the case of a pin support $\delta \ell_r$ must be zero, and when rotation is prevented $\delta\theta(\ell_r) = 0$, and it is such conditions which modify the treatment of the previous section.

The limiting case of linear splines corresponds to beam theory when the deflections y from the unstrained spline, considered to lie along the x-axis, are such that $|dy/dx| \ll 1$. To sufficient accuracy, x can replace arc length s and the support forces can be considered to act in the y direction, and then the longitudinal force $\bar{P}$ is zero throughout. From (10) the differential equation for the spline then takes the form

$$\frac{d^2\kappa}{dx^2} = 0 \quad , \tag{38}$$

with the linear approximation

$$\kappa = \frac{d^2y}{dx^2} \quad . \tag{39}$$

This immediately leads to piecewise cubic polynomials for y as a function of x . The variational principle for linear splines is that they minimize (1).

Schweikert [11] has treated linear splines under tension, in which end supports supply a positive longitudinal force $\bar{P}$, which is constant throughout the beam, for freely sliding constraints. By linearization of (10) it follows that

$$\frac{d^4y}{dx^4} = \bar{P}\frac{d^2y}{dx^2} \quad ,$$

so the solution between successive supports takes the form

$$y = c_1 + C_2 x + C_3 \cosh(\sigma x) + c_4 \sinh(\sigma x) ,$$

where $\sigma = \sqrt{\bar{P}}$. One reason for introducing tension is to remove extraneous points of inflection of the interpolating spline curve. The variational principle for linear splines under a given tension $\bar{P}$ is that they minimize the total energy of the system, which leads to minimizing

$$\int_{x_0}^{x_n} [f''(x)^2 + \bar{P} f'(x)^2]\, dx$$

among all functions f that satisfy the constraints and have continuous second derivatives. One could also study nonlinear open splines under tension.

The theory presented heretofore leads to some interesting characteristics for particular situations. For example, both for the open spline with minimum constraints depicted in Figure 2, and the closed spline of optimum length,.(29) requires that $\bar{P}$ be zero or negative, and zero only where the spline arc is straight. Thus, whatever the geometry of the curve being fitted, tensile resultant longitudinal forces will never occur (unless they are imposed at the ends).

Consider now fitting a closed spline through the vertices of an equilateral triangle. If the spline is bent into a circle, we see from (3) that M is constant, whence from (5) $S \equiv 0$, and from (6) $P \equiv 0$. Hence (29) is violated by a circle. To satisfy (29) some additional arc length must be added to produce a compressive force P . The "optimum" spline will take the form illustrated in Figure 4. A qualitative understanding of this deduction can be achieved by noting that increasing the arc length for a given angle of bend tends to reduce the contribution to the integral (14), just as adding large loops to a spline configuration permits the integral (14) to be reduced towards zero, as mentioned in [3]. With radius R , the $1/R^2$ of the integrand dominates the $2\pi R$ of the total arc length, for increasing R .However, for a fixed arc length and total angle of bend ($\int$ Kds) , the contribution to (14) is a minimum when κ is constant. Increasing the arc length of the spline in Figure 4 from the circle configuration causes a variation in curvature which tends to increase the integral, offsetting the reduction associated with increase in arc length. The latter dominates initially, to yield an optimum fit illustrated in Figure 4.

This example permits an assessment of the interpolation strategy expressed in (2), since one might regard the circumscribing circle as providing a more natural fit through the vertices. The advantage of increasing the arc length in reducing the integral (14) is the feature which leads away from the constant-curvature circle. Inhibition of such a tendency can be achieved by imposing a penalty on increase in arc length, for example, by replacing (2) by

$$\delta \int_{\ell_0}^{\ell_n} (\kappa^2 + k)ds = 0 . \tag{40}$$

Equation (37), and hence (29), must then be replaced by

$$\frac{\kappa^2}{2} + \bar{P} = \frac{k}{2} , \tag{41}$$

so that for this simple case, choosing

$$k = \kappa_0^2 , \tag{42}$$

where κ_0 is the curvature of the circumscribing circle, yields that circle as the optimum fit according to (40). Whether such an approach could be generalized is an open question.

If for a closed spline loop passing through prescribed points, the arc length is slightly shorter or longer than the optimum length given by (29), the integral (14) will be larger than for the optimum case. For each of these problems, with fixed arc length, (2) is satisfied by the curve form assumed by the spline. An illustrative example is given in Figure 5. For the shorter spline loop

$$\frac{\kappa^2}{2} + \bar{P} > 0 \qquad (43)$$

and for the longer one

$$\frac{\kappa^2}{2} + \bar{P} < 0 \quad . \qquad (44)$$

These conditions will change the constant c_{r1} in the governing differential equation (12), which will apply throughout the spline with constant c_{r1} if the supports are freely sliding and rotating.

This paper has treated the global problem of spline geometry. The computation of spline functions to approximate the spline configurations considered here has not been discussed in this paper, and constitutes a challenging problem in numerical analysis. For the open spline, the curvature at the first support is zero, so that only the angle need be determined if an initial-value approach (the so-called "shooting method") is used for integration of the spline differential equation problem. In the general closed spline case, both angle and curvature at a support must be selected for an initial-value approach, thus posing a more cumbersome problem. For the problem of the equilateral triangle, symmetry can be used to reduce the complexity of the general case. However, the work of Woodford [12] makes it seem unlikely that shooting is a good way to compute splines.

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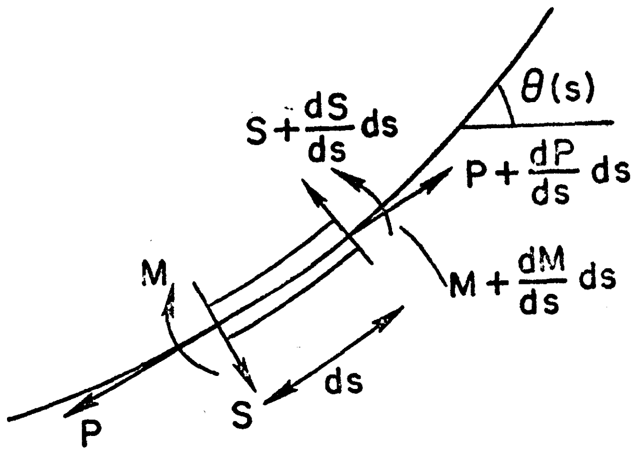

Figure 1. Forces and moments on a beam element.

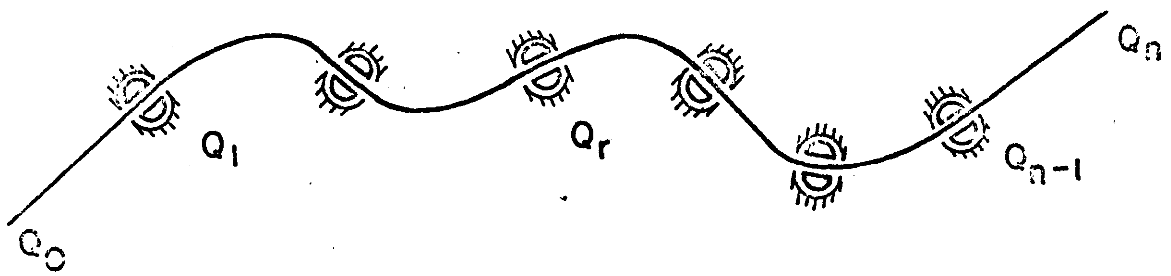

Figure 2. Spline passing through supports.

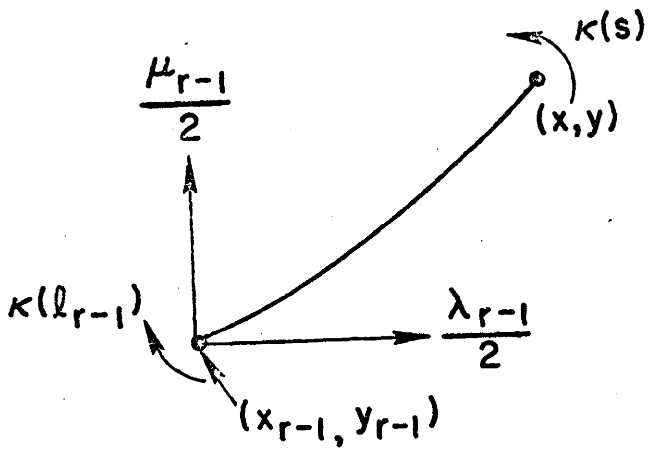

Figure 3. 'Forces and moments on a spline arc.

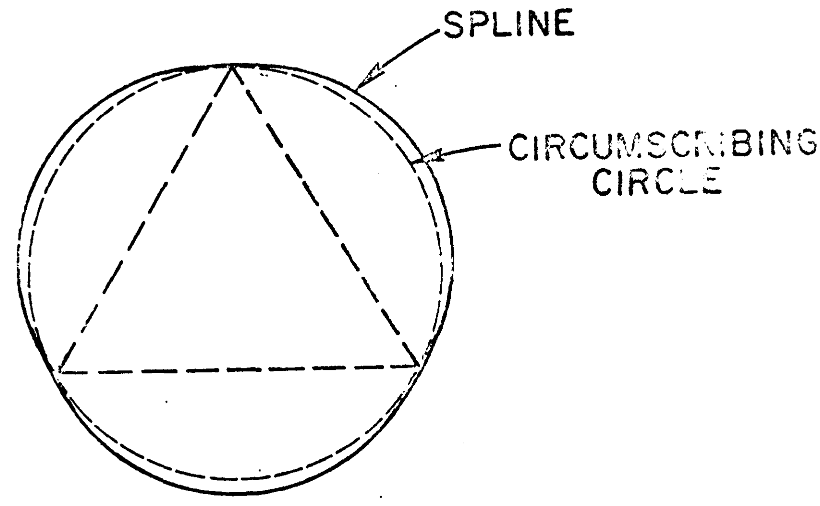

Figure 4. Spline fitted through the vertices of an equilateral triangle.

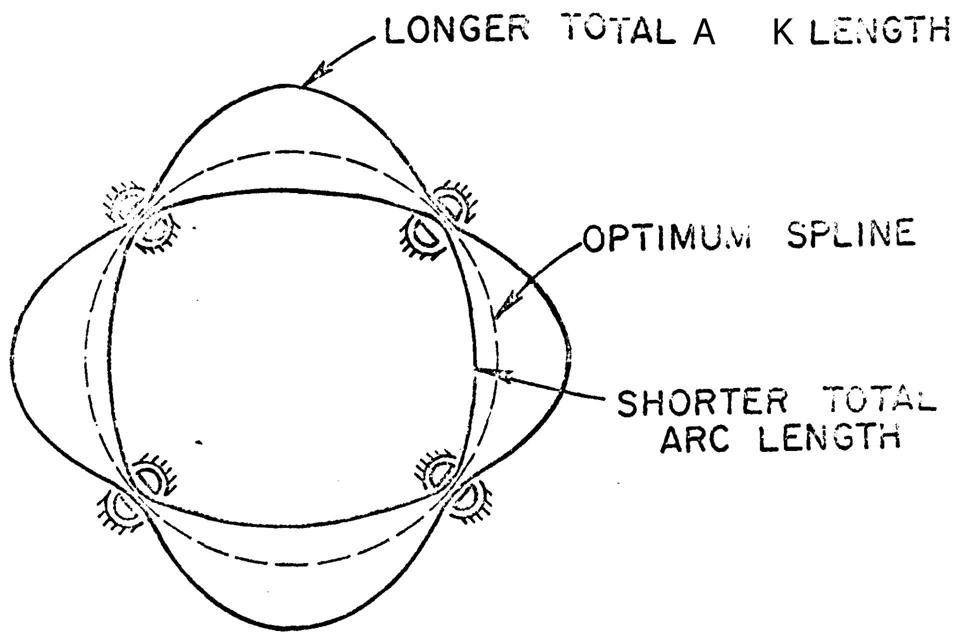

Figure 5. Closed splines with differing arc lengths.